Derivation Supplement — sharp-threshold modified gravity and Gaia wide binaries

Luc Côté — Zenodo deposit, DOI 10.5281/zenodo.22801278 — Theory of the Intersection of Fields

Origin of the four quantities used in the Letter, and derivation of the bump

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Purpose. This supplement derives the quantities the Letter uses — the presence fraction f = 1/16, the threshold rt = √A, the extinction factor and its 2π, the many-body superposition rule, and the bump above the threshold — from a single postulate: a tension interface whose static equilibrium is σ·Tr(K) = ρE. It presents no other part of the underlying framework. Each quantity is obtained with no fitted parameter; none is calibrated on binary data.

1. The presence fraction f = 1/16

What the presence fraction is. The source does not contribute to the interface Σ permanently: in each field, it spends part of its phase on Σ — present — and part outside Σ — absent. The presence fraction f measures the proportion of phase during which it is effectively present on Σ.

Where the value 1/16 comes from. An excitation leaves the interface through a half-turn, and it has four directions in which to do so: the three of the interface itself and its normal, the interface being of codimension 1. This counting bears on the exits the interface offers, not on what it separates — the three fields of the Letter belong to the second counting, independent of the first. A half-turn in one direction halves the presence, the projection of a uniform rotation giving exactly one half; after a half-turn in each of the four, there is no way out left. The minimum presence of an object still present on the interface is therefore

The coefficient 4/3. Both factors read off the shape equation in spherical geometry. The operator there splits into a radial term and a front term, σ·(2/r)·∂_r, whose 2/r is the curvature trace of the fronts. The interface being three-dimensional, that trace counts three directions around a spherical source: two angular, which form the 2/r, and one radial. The front term therefore carries two directions out of three — hence the two thirds, the missing third being the radial direction. The factor two comes from the cross term: the source as seen by the interface is quadratic in the presence amplitude, and two branches share the front — the vacuum branch, carrying the Laplace pressure σ·(2/r), and the fold branch, carrying σ·(2/r)·|∂_r u|. For the Newtonian well the two pieces cancel exactly and only their cross term survives, local to the front: it is the 2ab of $(a + b)^2$. The product, (2)·(2/3) = 4/3, admits no free ingredient. What remains open is writing this structure from the complete crossing action, which would add no ingredient but would close the formal chain.

$$f = (1/2)^4 = 1/16 \, .$$

This is a counting property, not an adjustment. The number 1/16 has no degree of freedom: it is fixed once the four directions are counted, and the factor ½ per direction is not posited but derived — a side is one sense of the normal, leaving through it is a half-turn, and the projection of a uniform rotation onto the interface is exactly one half.

2. The threshold r_t = √A: the front

The transition separates two regimes according to the comparison between the variation of the perturbation and the curvature of the background. Above the threshold, the gradient of the perturbation varies fast against the background curvature: the latter remains invisible and the dynamics is Newtonian. Below, the perturbation varies more slowly than the background curves, and the expansion around a flat background loses its meaning — this is where the dynamics deviates.

The radius where the gradient of the metric perturbation crosses the background curvature — the crossing radius — is written with the framework's quantities alone, the object's energy E and the tension σ:

$$r_t = \sqrt{A} = \sqrt{(E/(12\pi\sigma))} \, ,$$

where $A = E/(12\pi\sigma)$ is the shape constant of the source's proper field. No parameter is adjusted. The threshold is written with the object's quantities alone and applies, with the same coefficient, at any scale where a tension source meets the background; for a stellar pair, it falls at several thousand astronomical units.

Mass dependence. Since E grows with the object's mass, the threshold grows as $\sqrt{M}$. It is therefore not universal: the position of the step shifts from one pair to another according to its mass (Figure 2 of the Letter).

Acceleration at the threshold. The Newtonian acceleration evaluated at the front is independent of the source: the mass cancels against the energy, $GM/r_t^2 = 12\pi\sigma \cdot GM/E = 12\pi G\sigma/c^2 = c^2k \approx 1.15\times10^{-9}\ \mathrm{m\cdot s^{-2}}$. This is the unique acceleration scale at which the transition operates, whatever the mass. The crossing radius $\sqrt{(2A)}$, where the acceleration is half that value — $6\pi G\sigma/c^2 \approx 5.8\times10^{-10}\ \mathrm{m\cdot s^{-2}}$, which can also be written $\sqrt{(3\pi G\rho_\Lambda)}$ —, is a different place and does not mark the turnover; the two must not be conflated. Since the tension σ that fixes it is inherited from the vacuum energy density, and not from an acceleration scale, its coincidence with the observed threshold of the binaries is a consequence of the cosmological value of $\rho\Lambda$, not a calibration.

Origin of σ. The interface tension is fixed by two independently measured quantities, the vacuum energy density $\rho\Lambda$ and Newton's constant G, by inversion of the relation $G = c^4\rho\Lambda/(12\pi\sigma^2)$: $\sigma = \sqrt{(c^4\rho\Lambda/12\pi G)} = 4.13\times10^{16}\ \mathrm{J\cdot m^{-2}}$. No acceleration scale enters this determination. The closure curvature $k = \rho\Lambda/\sigma$ is a derived quantity. The value of σ is therefore independent of the wide-binary data.

3. The extinction factor and its 2π

Above the threshold, the crossing coordinate ceases to oscillate freely and becomes evanescent. Its behaviour is governed by the phase equation of the mode whose effective potential is the threshold itself: below the threshold, the mode is oscillatory (phase equipartition is recovered, and gives the 1/16 of Section 1); above, it is evanescent.

The phase equation. Writing φ for the crossing angle and ψ for the mode amplitude, the mode obeys a phase equation whose effective potential is the threshold itself,

$$d^2\psi/d\varphi^2 = (x - 1)\cdot\psi\ ,$$

with two regimes across $x = 1$. For $x < 1$ the right-hand side is negative and the mode is oscillatory, $\psi \sim \cos(\sqrt{(1-x)}\cdot\varphi)$, $\sin(\sqrt{(1-x)}\cdot\varphi)$ — phase equipartition, which yields the 1/16 of Section 1. For $x > 1$ the right-hand side is positive and the mode is evanescent, $\psi \sim e^{(-\sqrt{(x-1)}\cdot\varphi)}$. The crossing coordinate φ is an angle of period 2π (one full turn), so the presence amplitude over a complete turn decays as $e^{(-2\pi\sqrt{(x-1)})}$: the extinction factor and its 2π are read directly from the mode, not posited.

Origin of the 2π. The crossing coordinate is an angle, whose natural period is one full turn. The factor 2π appearing in the exponent is that period — the angle closing upon itself. The absence fraction above the threshold thus decreases as

$$1 - f_1 = \tfrac{1}{2}\cdot e^{(-2\pi\sqrt{(x-1)})}\ ,$$

continuous at the threshold $x = 1$. The 2π is not adjusted: it is the period of the crossing angle, imposed by the geometry of the mode.

The coefficient 0.323. The additional slope below the threshold is fixed by a chain — amplitude interference, scale tensions, shape equation — whose result, in this sector of the framework, reads $(2\cdot2/3)\cdot\sqrt{(f(1-f))}\cdot\sqrt{A/r}$. The factor $2\cdot(2/3) = 4/3$ combines a factor 2 from the interference of the two amplitudes and the 2/3 of the shape equation (the missing third in spherical geometry); it contains no free parameter. The dynamical ratio then reads

$$g_{obs}/g_N = 1 + (4/3)\cdot\sqrt{(f(1-f))}/\sqrt{x} = 1 + 0.323/\sqrt{x}\ ,$$

the number 0.323 being $(4/3)\cdot\sqrt{(f(1-f))}$ evaluated at $f = 1/16$. Once the four directions are counted, and the factor 4/3 fixed by the chain above, the slope below the threshold has no degree of freedom: the comparison of this value with binary data is a test, not a fit.

4. Many-body superposition

The effect of a local field on a stellar pair requires composing two sources. The framework does this exactly, by a structural property of its field equation.

10.5281/zenodo.22801278

Additive displacements. The field equation $\sigma \cdot \mathrm{Tr}(K) = \rho_E$ is linear in the displacement u: if two sources each produce a solution, their sum is a solution. The displacements add, $u_1 + u_2$.

Multiplicative clock rates. But the clock rate — the temporal factor that carries the dynamical effect — is exponential in that same displacement. For two bodies:

$$e^{2k(u_1+u_2)} = e^{2k\cdot u_1}\cdot e^{2k\cdot u_2} .$$

The clock rates multiply, exactly as they do between layers of a single well. The composition principle that fixes the second order thus extends to the many-body problem, exactly solvable at the field level, with no external-field-effect parameter. In other words, in the regime considered the displacement u obeys a linear equation while the metric observable depends exponentially on u: the nonlinearity is in the dressing, not in the field equation.

Consequence — the plateau. For a pair immersed in a background field, the background displacement uloc adds to that of the pair before entering the exponential; at large separation, where the pair's proper field fades, only the background clock rate survives. The ratio then tends toward a plateau fixed by the local baryonic slope alone $x_{loc} = \bar{g}\cdot c^2/(12\pi G\sigma) = \bar{g}/(c^2 k)$:

$$\gamma(s\to\infty) = 1 + (4/3)\cdot\sqrt{f(1-f)}/\sqrt{x_{loc}} , \quad \text{with } f \text{ evaluated at } x_{loc} .$$

The height of the plateau is therefore not a parameter: it is fixed by the numerical value of the local baryonic slope $\bar{g}$, which published models place in the interval $\bar{g} \approx (1.5\text{–}1.9)\times 10^{-10}\ \mathrm{m\cdot s^{-2}}$, whence $\gamma(\infty) \approx 1.80\text{–}1.90$. Its derivation contains no quantity fitted on the binaries. The dependence of the plateau on the local value of the background field is a prediction: the plateau rises where the local baryonic slope is weaker.

5. The bump above the threshold: derivation and projection

The Letter notes a fine bump of the ratio just above the threshold. This bump admits a closed derivation, with no parameter, from the same chain as the slope below the threshold; its peak height is an exact fraction, and its survival under projection onto the observables is estimated by the same Monte Carlo draw as the figures.

The unified form. The coefficient (4/3) and the interference term $\sqrt{f(1-f)}$ of Section 3 do not depend on the regime; only the presence fraction f changes. The ratio of the two scale tensions — that of the fold, $\sigma(2u/r^2)$, and that of the vacuum, $\sigma(2/r)$ — equals exactly the Newtonian slope, $\sigma(2u/r^2)/\sigma(2/r) = u/r = x$: the variable x is that internal ratio. Below the threshold, the vacuum dominates and equipartition fixes $f = 1/16$; above, the fold occupies the front and the crossing coordinate becomes evanescent, so that the presence of EACH of the four directions rises according to

$$f_1(x) = 1 - \tfrac{1}{2}\cdot e^{-2\pi\sqrt{x-1}} \quad (x \geq 1), \quad \text{the total presence being} \quad f = f_1^4 .$$

The dynamical ratio thus reads, on both sides of the threshold, in a single form,

$$g_{obs}/g_N = 1 + (4/3)\cdot\sqrt{f(1-f)}/\sqrt{x} ,$$

with $f = 1/16$ below the threshold and $f = f(x)$ above. There is no new ingredient: the same chain governs both regimes.

The match at the threshold, exact. At the threshold f_1 equals ½, so the total presence is $(\tfrac{1}{2})^4 = 1/16$ — exactly the equipartition value that holds below. The ratio is therefore continuous at the threshold, with no adjustment; it is the counting of the four directions, the same that gives the 1/16, which secures the match. The maximum of the interference, $f = \tfrac{1}{2}$, is reached slightly above, at $x = 1.03$, where the ratio peaks at

$$\gamma_{peak} = 1 + (4/3)\cdot(\tfrac{1}{2})/\sqrt{1.03} = 1.656,$$

with no degree of freedom — imposed by $f = \tfrac{1}{2}$ and the coefficient 4/3 already fixed below the threshold. The peak then descends toward the plateau as the presence saturates ($f \to 1$) and the interference dies out. In separation, the bump occupies a few per cent of rt, just inside the threshold.

Survival under projection. It remains to know whether this fine structure survives the passage into the observable plane. We project it by the same Monte Carlo draw as the figures of the Letter: $N = 6\times 10^5$ systems with log-uniform semi-major axes over [0.8; 40] kAU (kilo-astronomical units, 10^3 AU), thermal eccentricities $p(e) = 2e$, sampled orbital phases (Kepler's equation), isotropic line of sight; the background-field effect is composed exactly, $x = x_{proper} + x_{loc}$, the slopes adding because the displacements add. For a total mass of 1.5 $M\odot$ (rt ≈ 3.9 kAU), the projected ratio with the bump exceeds the smoothed ratio by

10.5281/zenodo.22801278

Δγ ≈ +0.15 (2–2.5 kAU), +0.19 (2.5–3 kAU),

the gap vanishing beyond 3 kAU, the interference maximum having been crossed. The projection above uses Kepler's equation, which assumes closed orbits — which this law does not give, since it makes the orbit precess by 50 to 60° per revolution. The check has been made: drawing the radius with the time density of the potential itself, dt ∝ dr/|v_r|, leaves the shoulder unchanged to within 0.003 in every bin. The reason is that precession shifts the axis of the orbit without appreciably altering the distribution of radii — a rosette and an ellipse with the same pericentre and apocentre occupy the same radii, in a different order — and the projected shoulder depends only on that distribution and on the viewing angle. The bump is therefore not erased by projection: it leaves an observable shoulder in the 3–4 kAU window, visible in Figure 1 of the Letter just before the rise toward the plateau. This signature — a narrow shoulder, indexed on the threshold rt and hence shifted by the mass of the pair — constitutes an additional prediction, distinct from the plateau, and requires a sample finely resolving the threshold zone.

Summary

The quantities of the Letter all follow from the single postulate of a tension interface: f = 1/16 by counting the four exit directions; rt = √A as the front, the place where the Newtonian acceleration reaches $12\pi G\sigma/c^2$; the 2π as the period of the crossing angle in the phase equation of the mode; multiplicative superposition as a consequence of the linearity of the field equation in the displacement; and the bump above the threshold as the same interference chain, with the exact match at the threshold from the counting of the four directions, and a peak γ = 1.656 whose shoulder at 3–4 kAU survives projection. None is calibrated on wide-binary data: the comparison of the Letter with Gaia is therefore a test, not a fit.

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